A General Analytic Model for Demand Responsive Transport with Heterogeneous Demand

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ABSTRACT
This paper presents a general continuous approximation model for DRT corridors that takes into account the spatial variability of all parameters. The goal is to enable a more precise definition of the problem while retaining the benefits of continuous closed-formula models. Some of the effects include variable-width corridors, non-homogeneous demand, changing street networks, and different operations at different points along the corridor. Furthermore, a detailed definition of route dynamics allows for the study of the effects of the street network pattern (circuitry) on DRT routing.

The model is then used and adapted to answer three questions: (a) what is the effect of spatially heterogeneous demand and how the service can adapt to it; (b) what are the reasons and circumstances under which transit agencies may financially justify substituting bus conventional services for DRTs; and (c) how can an increase in input precision improve the results of previous models and aid in the more accurate design of DRT systems. Furthermore, three on-going DRT use cases in the Barcelona metropolitan region (Torre Baró, Vallirana, and Cervelló) are used to validate each of the findings.
1. Introduction

Public transportation is one of the core services of modern societies. New mobility laws are evolving into guaranteeing accessibility and freedom of movement for all citizens, as well as reducing the externalities caused by private motorized vehicles. To achieve this goal in the most cost-effective way, demand for similar journeys must be aggregated in a single vehicle, resulting in a scalable collective transportation system. If the demand is served with individual vehicles, economies of scale are lost and both democratizing and environmental effects of collective transportation disappear.

Unfortunately, these principles are hard to apply in suburban and other low-density areas. Without sufficient demand, even the lowest capacity traditional service (small bus) becomes inefficient for both agency (not enough revenue) and users (long waiting and travel times due to infrequent and circuitous routes). These areas are particularly vulnerable to the commonly known vicious circle of public transportation, in which low demand leads to low-quality, unappealing service, which further reduces demand. In the end, public transportation is used only by those who do not have access to private vehicles (young, elderly, and lower-income population) and is maintained/subsidized to ensure these groups' bare minimum mobility rights (Vuchic, 1999).

Within this context, demand responsive transportation (DRT) emerges as a more flexible scheme capable of adapting to low density zones, resulting in an effective and efficient service. For the agency, knowing the demand beforehand can save long detours to empty bus stops or even full empty cycles. For the users, not only they can receive a more personalised service but reduce their walking and travelling time. This flexibility comes at the expense of the scalability that defines conventional transportation. If a DRT bus starts filling up, detours to pick up and drop off passengers become unsustainably long and the quality of service drops down. Therefore, DRTs are not an easy service to design or operate. The operator wants to increase their revenue by filling the vehicle, but if occupancy is high, the user is penalised and demand is lost. Furthermore, it all usually occurs in areas where the private vehicle alternative is quite appealing and where acquired demand is highly sensitive to quality of service.

Historically, DRT began as a service tailored to specific segments of known demand, such as people with disabilities and non-urgent patient transportation or scholar/work-related trips. Technological advancements over the last decade have resulted in the emergence of general public DRTs, which provide either a premium service or the best value public transportation for low-demand zones (Aberdeen et al., 2006). Unfortunately, commercial DRTs have a high failure rate, with 50% of them ending service in less than 7 years and 40% in less than 3 (Currie & Fournier, 2020), with examples such as Kutsuplus in Helsinki or Chariot in San Francisco.

For that reason, DRT design and operation strategies have been extensively studied in literature both for target groups and top quality services. The design of this kind of services is somehow linked to solving a dial-a-ride problem (DARP) for the studied scenario, so one must face the complexity of VRPs in order to obtain the service main characteristics (travel and waiting times, fleet size, total route distance, etc.). To that end, literature has taken different approaches, which can be roughly divided between those that focus on simulation and the use discrete mathematics to solve the DARP and those that propose an approximate analytical solution in order to obtain a less precise but more insightful closed formulation.

The first methodological approach focuses on obtaining a more precise solution for DRT problems. DARP is attacked fully and therefore detailed inputs and enough computation time are required. A series of methods have been proposed, including traditional algorithms like branch-
and-cut (Cordeau, 2006; Dikas & Minis, 2014) and heuristic methods such as tabu search (Cordeau & Laporte, 2003). Dynamic optimization is also proposed, with insertion heuristics like the ones in (Luo & Schonfeld, 2011) and multi period VRP solutions as in (Albareda-Sambola et al., 2014). A more detailed insight on how these methodologies apply to the solution of the DARP can be found in (Cordeau & Laporte, 2007).

On the other hand, in an attempt to construct more insightful models, the second approach (and the one followed in this paper) simplifies reality and simulates the problem by means of approximations that fulfill the widely accepted principles of geometric probability and queuing theory. One of the first analytical models for DRT was presented in (Daganzo, 1978) which studied the average waiting and riding time of three different dispatching algorithms. Since then, several authors have studied flexible systems under different hypothesis and using diverse methodologies. (Chandra & Quadrifoglio, 2013; Quadrifoglio & Li, 2009) use continuous approximations to analyse DRT as a feeder line. (Kim & Schonfeld, 2012) (extended for a whole region in (Kim et al., 2018)) study the connection of local regions to a main terminal using a service that combines a DRT zone and a haul direct route. (Nourbakhsh & Ouyang, 2012) combine corridors of flexible buses to serve a large squared area, mimicking the conventional bus transfer network proposed in (Daganzo, 2010). (Daganzo & Ouyang, 2019) give a wider insight on shared transportation analysing DRT with a queuing/dynamic states model, which resembles more the modelling of taxi systems. Finally, regarding inserting DRT as a part of a wider conventional network, research has been done in (Aldaihani et al., 2004) and (Liu & Ouyang, 2021).

Most of these models rely on continuous approximations due to the simplicity of the required inputs and their ease to produce analytic insights. These hypotheses work best for large homogeneous areas and end being quite reliable. Unfortunately, the characteristics of potential DRT locations usually differ from this homogeneity. Suburban zones tend to be irregular in shape, with a disjoined street network and changing characteristics, and the small number of trips to be served increases input variability both along time and space.

Therefore, this paper proposes a model that accounts for spatial variability in all inputs without losing the analytical form of continuous approximation models. In a static corridor-like scheme; demand, width, street spacing and all service characteristics can vary along the longitudinal direction, resulting in a much more detailed definition of the problem. The increased precision in the inputs raises the reliability of the results and the flexible deployment in the service characteristics allows the service adapt to the changing conditions along the corridor. In fact, formulation can model both conventional bus and DRT services, and even a combination of the two. Moreover, new routing strategies are considered and street network is incorporated into the model either as a closed formula derived from idealised grids or via the real circuitry metric. All in all, a generalist and versatile model is presented that encompasses the majority of corridor-level situations.

Formulation is based on (Daganzo, 1984), (Nourbakhsh & Ouyang, 2012) and (Estrada et al., 2021) and has been constructed taking a step back from the cited models, removing some of their hypotheses and moving towards generality. However, the goal of this step back is not only to relax previous services assumptions, but also to present a generalist framework that can be easily used and adapted to any routing scheme that the modeller wishes to test.

The paper is structured as follows. Section 2 formulates the general model. Sections 3 to 5 focus on assessing the three use cases. Finally section 6 presents the final conclusions. Annexes also
show the deduction of the general model and gather some extra insights on the inclusion of some elements inside it.

2. Modelling framework

2.1 Problem statement

This paper considers a public transportation corridor with vehicles running at a constant headway \((H)\) from \(x = 0\) to \(x = L\) and back. The service characteristics are defined with the following hypotheses:

1. The service is provided inside the area defined by a constant length \((L)\) along the X axis and a variable width \((w(x))\), function of \(x\). The area is symmetric respect the X axis (Figure 1).

![Figure 1: Corridor with variable width.](image1)

2. Similarly to (Daganzo, 2010), the demand density \((\lambda)\), i.e. the number of trips generated per hour per unit area, is considered to be variable along the X axis but homogeneous along the Y axis (Figure 2). Trip origins and destinations are independently distributed in the region according to a spatial Poisson process with average \(\lambda\) trips in every \(dx\).

![Figure 2: Corridor with demand density distribution.](image2)

3. The local street network is an orthogonal grid aligned with the X-Y axis with a street spacing \(D_0(x)\). Street spacing can also vary for different regions along the X axis (Figure 3). Therefore, all travelled distances, by foot or inside a vehicle, are computed using Manhattan metric \(L_1\) (being 1 and 2 origin and destination points, the distance between them is \(D = |x_2 - x_1| + |y_2 - y_1|\)).
Within this self-contained corridor, the demand is served with a unique direct service that can combine two operation strategies:

a) Conventional bus (service 0), understood as a traditional fixed line with the ability to skip empty stops. For narrow corridors, it may only travel along the central X axis. Nevertheless, for wider corridors, if a maximum access walking distance (r) is exceeded, the conventional bus line needs to swap up and down to guarantee minimum accessibility standards.

b) DRT (service 1) is understood as a fully door-to-door direct service. In this case, accessibility is not a limiting factor but, as demonstrated in (Daganzo, 1984), some virtual bands may be defined in order to approach the optimal close-to-minimum total trip length for wide corridors.

The possible studied services can be visualized in Figure 4, showing also the different route shapes depending on the width of the corridor (narrow/wide).

4. Formulation allows for conventional and DRT behaviour to be combined in a single bus corridor, with the same vehicles performing a different service depending on the point of the route where they are. This spatial combination is controlled by the Boolean alpha variable (α(x)), dependant on x and with a value of 0 for the conventional sections and 1 for the DRT zones (Figure 5).
Figure 5: Corridor with a combination of conventional and DRT services.

5. For the sections served with a conventional bus system (service 0), stop spacing \( s(x) \) is defined along the route length, not the corridor length (Figure 6). It can also vary for different points of the route.

Figure 6: Stop spacing \( s(x) \).

6. Vehicles have a fixed capacity of \( C \) passengers per vehicle. Maximum occupation \( O_{\text{max}} \) cannot exceed capacity.

Overall, the proposed services and their associated models represent two of the most common schemes for low-demand corridors in a straightforward CA style, with some of its common hypotheses but sufficient detail to understand the services main dynamics. Also, the variability of both inputs and enables the study of longer corridors that pass through different neighbourhoods with different characteristics without losing too much information during the parameter definition phase.

### 2.2 Horizontal circuitry

In order to aggregate 2D dynamics along a one-dimensional corridor, we introduce the variable called horizontal circuitry \( k(x) \). Analogously to traditional circuitry, it measures the directness of trips and thus the efficiency of the transportation service.

Horizontal circuitry is defined as the average quotient between the real travelled distance and the horizontal distance for any pair of points (Equation (1)). As characteristics change along the X axis, horizontal circuitry does so, capturing differences on all network, demand and 2D operations into a single variable.
This variable essentially increases the travelled length in relation to the horizontal length, including all information regarding the various routing schemes and street network characteristics. Furthermore, in addition to the phenomena discussed in this paper, this parameter can be easily changed to represent different operation schemes.

Several formulations for \( k(x) \) are derived from this point, representing close-to-optimal routes for both conventional and DRT services depending on the effects to include; and a method to define it as an input parameter is proposed.

### 2.2.1 Horizontal circuitry in conventional bus (service 0)

As seen in Figure 4, the conventional bus only travels horizontally in the middle of the service area, without any deviation, when the corridor is narrow. Therefore, the horizontal circuitry for this case is the trivial:

\[
k(x) = 1
\]  

Nevertheless, in the more general case of wide corridors, it might happen that buses need to sweep the area up and down in order to fulfill accessibility standards. Imagine that administration sets any kind of restriction on accessibility/coverage of the line, e.g. a maximum walking distance \( r \) from any point to a stop. Depending of stop spacing \( s(x) \), there is a maximum corridor width in which the bus is able to operate and, if exceeded, vertical bands of that width need to be followed. For the said general case, band width \( w^*(x) \) is computed as:

\[
w^*(x) = \begin{cases} 
w(x) & \text{if } w(x) \leq 2r - s(x) \\
(2r - s(x)) & \text{if } w(x) > 2r - s(x)
\end{cases}
\]

Note also that for cases where \( s \geq 2r \) there is not any feasible solution, therefore it must be set as a boundary condition of the optimization problem. Horizontal circuitry can then be defined as following:

\[
k(x) = 1 + \frac{w(x) - w^*(x)}{w^*(x)} = \frac{w(x)}{w^*(x)}
\]

### 2.2.2 Horizontal circuitry in DRT (service 1)

The part of the trip the user used to walk in the aforementioned Service 0 is now travelled inside the vehicle in DRT systems. The proposed operation scheme consists on visiting the points in horizontal order and “sweeping the area up and down” to do so. Therefore, a transversal vertical component is added to total travelled length, which for a vertically uniformly distributed set of points is, on average, \( w/3 \). We can include this effect for a narrow corridor as:

\[
k(x) = 1 + \frac{w(x)}{3} \cdot \lambda(x) w(x) H
\]

When considering also wider corridors, an optimal band width is computed following (Daganzo, 1984). In that case, virtual bands are introduced to minimize total route length.

The optimal width of these bands is:
\[ w^*(x) = \frac{3}{\sqrt{H \cdot \lambda(x)}} \]  

(6)

And including the boundary condition that the optimal width cannot exceed the real width of the corridor:

\[
\begin{align*}
 w^*(x) = & \left\{ 
 \begin{array}{ll}
     w(x) & \text{if } \frac{3}{\sqrt{H \lambda(x)}} \geq w(x) \\
     \frac{3}{H \lambda(x)} & \text{if } \frac{3}{H \lambda(x)} < w(x)
 \end{array}
 \right.
\end{align*}
\]  

(7)

Finally, circuitry for this case is defined similarly, just adding the perpendicular swap inside the stripes:

\[
k(x) = \frac{w(x)}{w^*(x)} + \frac{w^*(x)}{3} \cdot \lambda(x)w(x)H
\]  

(8)

### 2.2.3 Taking into consideration street network without empirical data

Real networks usually differ from ideal, on-the-map measurements. In the previous expressions of circuitry, the use of Manhattan metric already accounted for some of this increase of the route, but extra effects can be added, especially in the DRT routing schemes.

(Nourbakhsh & Ouyang, 2012) explored extra detours for orthogonal two-way street network. So, imagine the orthogonal network oriented along the X axis, and with both vertical and horizontal street spacing equal to \(D_s\). For most of the trips between DRT stops, the previous expressions accounting for total horizontal distance and extra vertical detour distance are enough, but in the case where two or more points are located in the same column of street blocks, the bus needs to travel an extra horizontal detour. This is a local effect that makes sense to take into account for small areas or big street blocks, but it is trivial to see that this extra component tends to zero for an infinite area. If we imagine that the vehicle cannot turn around in the middle of the street (needs to arrive to next intersection), the extra detour for every extra point is \(2D_s\). On the other hand, the probability of performing one stop in a certain column is \(D_s(x)\lambda(x)w(x)H\). Therefore we can slightly increase circuitry considering this effect like:

\[
k(x) = \frac{w(x)}{w^*(x)} + \frac{w^*(x)}{3} \cdot \lambda(x)w(x)H + \sum_{i=2}^{\infty} (i - 1) \cdot 2D_s(x) \cdot \left( \frac{D_s(x)\lambda(x)w^*(x)H}{\Lambda} \right)^i
\]  

(9)

Nevertheless, finding two-way grids is quite difficult in reality, so we can also formulate the extra detour that an orthogonal one-way street network requires. In this case, an extra detour happens for almost every trip between two points. On average, this extra distance has a value of \(2D_s\): one \(D_s\) to get from any point of the street to nearer crossroad (start and end of trip) and the other \(D_s\) as the average detour for the route between crossroads (extended in Annex 1). Therefore, certain operational strategies could potentially reduce the detour by limiting the places to board or alight.
from the vehicle. Anyway, for the general door-to-door case the formulation of circuitry can be revised to be:

\[
k(x) = \frac{w(x)}{w^*(x)} + \left(\frac{w^*(x)}{3} + 2D_3(x)\right) \cdot \lambda(x) w(x) H
\]  

(10)

2.2.4 Taking into consideration street network using empirical data

Now imagine that instead of an ideal Manhattan grid we are able to take some metrics from the real network of the area of implementation, specifically circuitry of the network (defined previously). To compute an aggregated metric, a big enough sample of \( n \) O/D pairs is taken and circuitry for every pair \( C_n \) is computed. The average value for circuitry is:

\[
\bar{C} = \frac{\sum_{i=1}^{n} C_i}{n}
\]  

(11)

It must be accounted that we are assuming that circuitry is independent from trip length but it has been shown that this is not the case (Giacomin & Levinson, 2015). Therefore, in the determination of this input value for the system, only routes with length in the range of the expected trip distance should be taken into account. It should be also verified that only the part of the network where the bus can travel is taken into account.

If computed from scratch, one could compute every individual circuitry measure as horizontal circuitry (dividing by horizontal distance instead of Euclidean distance). Nevertheless, if circuitry is obtained from literature or a general value is taken, transformation to horizontal circuitry can be done as:

\[
\bar{k} = \frac{\bar{C}}{\cos(\bar{\phi})}
\]  

(12)

Being \( \bar{\phi} \) the average angle between any pair of points and the X axis. This value can be computed for a rectangular area of \( L \cdot w \) and homogeneous demand as:

\[
\bar{\phi}(L, w) = \int_0^L \int_0^w \frac{xy}{Lw} \cdot \Phi(x, y) + \frac{(L - x)(w - y)}{Lw} \cdot \Phi(L - x, w - y) + \frac{x(w - y)}{Lw} \cdot \Phi(x, w - y) + \frac{(L - x)y}{Lw} \cdot \Phi(L - x, y) \, dx \, dy
\]  

(13)

Being \( \Phi(X, Y) \) the angle fixing the origin point, and a function defined as:

\[
\Phi(X, Y) = \frac{1}{XY} \int_0^X \int_0^Y \arctan \left( \frac{Y}{X} \right) \, dy \, dx
\]

\[
= \arctan \left( \frac{Y}{X} \right) + \frac{1}{4XY} (Y^2 - X^2) \ln(Y^2 + X^2) - \frac{Y}{2X} \ln(Y) + \frac{X}{2Y} \ln(X)
\]  

(14)

This value can serve as an approximation for less regular areas taking average width. Nevertheless, it is highly recommended to compute horizontal circuitry from scratch.
2.3 Costs of the service

The costs are divided into two categories: agency costs and user costs. The first includes fleet and travel distance costs, while the second consists on a monetary equivalent for total users’ door-to-door and waiting time. All costs are calculated for an average hour of system operation and follow the structure proposed by (Estrada et al., 2021).

2.3.1 Agency costs

Hourly agency costs are estimated as the sum of fleet costs and travelled distance costs, as seen in the following expression:

\[ Z_A = c_d \cdot Q + c_t \cdot M \]  

(15)

Being \( Z_A \) the hourly agency costs, \( Q \) the distance travelled in an hour by all the fleet, \( M \) the number of vehicles, \( c_d \) the distance unit costs and \( c_t \) the temporal unit costs. Unit costs are pre-set inputs of the system and expressions for \( Q \) and \( M \) are derived below.

To do so, the first variable to compute is horizontal circuitry \( k(x) \), which for the general case, can be expressed as:

\[ k(x) = \frac{w(x)}{w^*(x)} + \frac{w^*(x)}{3} \cdot \lambda(x)w(x)H \cdot \alpha(x) \]  

(16)

Therefore, integrating along the domain, the distance travelled by the fleet in an hour (\( Q \)) is:

\[ Q = \frac{2}{H} \int_0^L k(x) \, dx \]  

(17)

To follow up, an intermediate variable also needs to be defined: the number of total effective stops. This takes into account the possibility of the conventional bus to skip stops.

\[ S_e(x) = \begin{cases} \min \left( \frac{k(x)}{s(x)}, \lambda(x)w(x)H \right) & \text{for} \quad \alpha(x) = 0 \\ \lambda(x)w(x)H & \text{for} \quad \alpha(x) = 1 \end{cases} \]  

(18)

The total number of stops is computed as:

\[ S_e = \int_0^L S_e(x) \, dx \]  

(19)

At this point, commercial speed (\( v_c \)) can be computed, which includes the effects of the free flow trip, vehicle acceleration and deceleration, and passenger boarding time:

\[ \frac{1}{v_c(x)} = \frac{1}{v} + \frac{1}{k(x)} \left[ S_e(x) \cdot \frac{v}{a} + \lambda(x)w(x)H \cdot \tau' \right] \]  

(20)

With \( v \) being the bus cruising speed, \( a \) the acceleration and deceleration and \( \tau' \) the unit boarding and alighting time per user.

And again integrating along the domain, the size of the fleet (\( M \)) is:

\[ M = \left[ \frac{2}{H} \int_0^L \frac{k(x)}{v_c(x)} \, dx \right]^* \]  

(21)

2.3.2 User costs
User costs are estimated as the product of total door-to-door time by the average value of time ($\beta_T$). This door-to-door time is divided in access time ($A$), waiting time ($W$) and in-vehicle travel time ($IVTT$). Hourly costs are defined as:

$$Z_U = (A + W + IVTT) \cdot \beta_T \cdot \Lambda$$

(22)

Being $\Lambda$ the total passengers travelling in an hour:

$$\Lambda = \int_0^L \lambda(x)w(x) \, dx$$

(23)

And knowing that occupancy can be computed as:

$$O(x) = \frac{H}{\Lambda} \left( \int_0^x \lambda(x)w(x) \, dx \cdot \int_x^L \lambda(x)w(x) \, dx \right)$$

(24)

In vehicle travel time is easily derived as:

$$IVTT = \frac{2}{\Lambda H} \int_0^L O(x) \cdot \frac{k(x)}{v_c(x)} \, dx$$

(25)

Average access time is only different to zero for conventional case:

$$A = \frac{1}{2v_w \Lambda} \cdot \int_0^L \left( 1 - \alpha(x) \right) \cdot \lambda(x)w(x) \cdot [s(x) + w^*(x)] \, dx$$

(26)

Waiting time is derived from headway and modelled considering headway behaviour or schedule behaviour depending on the kind of service and the value of headway, in an approach similar to (Badia, 2016; Tirachini et al., 2010):

$$W = \frac{1}{\Lambda} \cdot \int_0^L \lambda(x)w(x) \cdot \left[ \left( 1 - \alpha(x) \right) \cdot \omega + \alpha(x) \cdot h_s \right] \, dx$$

(27)

$$\omega = \begin{cases} 
\frac{H}{2} & \text{if } H < H_s \\
h_s + f_s H & \text{if } H \geq H_s 
\end{cases}$$

(28)

Being $h_s$ the average time a passenger arrives before departure if they know the schedule, $H_s$ the threshold headway where a scheme based on headways is starting to be treated as a scheduled scheme by users and $f_s$ a growth factor smaller than 0.5 associated with the extra margin passengers use for higher headways, due to potential variability.

### 2.3.3 Optimization

Once all costs are computed, the system can be optimized to find the best configuration. In our case, total cost of the system is minimized considering both user and agency costs equally, as a measure of the total social welfare lost due to service operation. The general expression to optimize can be written as:

$$\min_{H, s(x), \alpha(x)} Z = Z_A + Z_U = c_A \cdot Q + c_t \cdot M + (A + W + IVTT) \cdot \beta_T \cdot \Lambda$$

(29)

s.t. \hspace{1cm} O(x) \leq C \hspace{1cm} for \hspace{1cm} \forall x \in [0, L]$$

(30)
Decision variables are the headway \((H)\), the kind of service as in the alpha coefficient \((\alpha(\chi))\) and spacing between stops \((s(\chi))\). Optimization for this problem is done by grid search defining and testing a finite list of \(H\), \(s(\chi)\) and \(\alpha(\chi)\) value combinations.

### 3. Heterogeneous demand density

#### 3.1 Compact formulation

As a first approximation of the problem, a theoretical case is derived with analytical closed formulas. The proposed problem has the following input parameters:

\[
w(\chi) = w \\
\lambda(\chi) = \lambda \cdot x \\
x \in [0, L]
\]

And the following decision variables profiles:

\[
H = H \\
s(\chi) = s \\
\alpha(\chi) = [0,1]
\]

Moreover, the following assumptions are done:

1. The problem occurs in a narrow corridor, which implies:

\[
w \leq \frac{3}{\sqrt{H\lambda L}} \text{ and } w \leq 2r - s
\]

2. The conventional service stops in all stops, which implies:

\[
S_e = \frac{1}{s} \text{ for } \alpha = 0
\]

3. The street network is bidirectional with small enough street blocks, which implies that no extra term will be needed for the expression of \(k(\chi)\) and therefore the problem becomes independent to street spacing \(D_s\).

This problem configuration is set with the objective to have a simple straightforward deduction of the outputs, which will be all finally stated as closed expressions dependant only on the scalar input parameters.

To start, as we assume a narrow corridor (assumption 1), there are not any bands to be defined and therefore the optimal band width \((w^*)\) will be:

\[
w^* = w
\]

Horizontal circuitry \((k(\chi))\) can therefore be expressed as (also under assumption 3):

\[
H \geq H_{\text{min}}, \quad 0 < s(\chi) < 2r
\]
\[ k = 1 \quad \text{for } \alpha = 0 \quad (36) \]

\[ k(x) = 1 + \frac{\lambda w H}{3} x \quad \text{for } \alpha = 1 \quad (37) \]

Regarding the agency costs, distance travelled by the fleet in an hour \((Q)\) is:

\[
Q = \frac{2}{H} \int_0^L k(x) \, dx = \frac{2L}{H} \quad \text{for } \alpha = 0
\]

\[
Q = \frac{2}{H} \int_0^L k(x) \, dx = \frac{2L}{H} \left[ L + \frac{\lambda w H L^2}{6} \right] \quad \text{for } \alpha = 1
\]

The number of effective stops, as the conventional bus cannot skip any stop, is expressed as:

\[
S_e = \frac{1}{s} \quad \text{for } \alpha = 0
\]

\[
S_e(x) = \lambda w H x \quad \text{for } \alpha = 1
\]

And the total number of stops is computed as:

\[
\bar{S}_e = \int_0^L S_e(x) \, dx = \frac{L}{s} \quad \text{for } \alpha = 0
\]

\[
\bar{S}_e = \int_0^L S_e(x) \, dx = \frac{\lambda w H L^2}{2} \quad \text{for } \alpha = 1
\]

At this point, commercial speed \(v_c\) can be computed:

\[
\frac{1}{v_c(x)} = \frac{1}{v} + \frac{1}{k(x)} \left[ S_e(x) \cdot \frac{v}{a} + \lambda(x) w(x) H \cdot \tau' \right]
\]

\[
= \frac{1}{v} + \frac{v}{a} + \lambda w H x \cdot \tau' \quad \text{for } \alpha = 0
\]

\[
\frac{1}{v_c(x)} = \frac{1}{v} + \frac{\lambda w H x}{3} \left[ \frac{\lambda w H x \cdot v}{a} + \lambda w H x' \right]
\]

\[
= \frac{1}{v} + \frac{3 \lambda w H x}{3 + \lambda w H x} \left[ \frac{v}{a} + \tau' \right] \quad \text{for } \alpha = 1
\]

And the size of the fleet \((M)\) is:

\[
M = \left[ \frac{2}{H} \int_0^L \frac{k(x)}{v_c(x)} \, dx \right]^+ = \frac{2}{H} \int_0^L \frac{1}{v} + \frac{1}{s} \cdot \frac{v}{a} + \lambda w H x \cdot \tau' \, dx
\]

\[
= \frac{2}{H} \left[ L + \frac{L}{v} \cdot \frac{v}{a} + \frac{\lambda w H L^2}{2} \cdot \tau \right] \quad \text{for } \alpha = 0
\]

\[
M = \left[ \frac{2}{H} \int_0^L \frac{k(x)}{v_c(x)} \, dx \right]^+ = \frac{2}{H} \int_0^L \left( 1 + \frac{\lambda w H}{3} x \right) \cdot \left( \frac{1}{v} + \frac{3 \lambda w H x}{3 + \lambda w H x} \cdot \frac{v}{a} + \tau' \right) \, dx
\]

\[
= \frac{2}{H} \int_0^L \left( 1 + \frac{\lambda w H}{3} x \right) \cdot \left( \frac{1}{v} + \frac{3 \lambda w H x}{3 + \lambda w H x} \cdot \frac{v}{a} + \tau' \right) \, dx
\]
\[
\frac{2}{H} \left( L + \frac{\lambda wH L^2}{6v} + \left( \frac{v}{a} + \tau' \right) \cdot \left( \frac{3\lambda w L^2}{2} - 6L + \frac{18}{\lambda w} \ln\left( \frac{\lambda w H L}{3} + 1 \right) \right) \right)
\]  

\text{for } \alpha = 1

For the user costs, the total passengers travelling in an hour:

\[
\Lambda = \int_0^L \lambda(x)w \, dx = \frac{\lambda w L^2}{2}
\]  

(48)

And therefore the occupancy can be computed as:

\[
O(x) = \frac{H}{\Lambda} \left( \int_0^x \lambda(x)w(x) \, dx \cdot \int_x^L \lambda(x)w(x) \, dx \right) = \frac{H\lambda w}{2L^2} \cdot \left( x^2L^2 - x^4 \right)
\]  

(49)

With maximum occupancy found at:

\[
\frac{dO(x)}{dx} = \frac{H\lambda w}{L^2} \cdot (xL^2 - 2x^3) = 0 \rightarrow \begin{cases} x = 0 \\ x = \pm \frac{\sqrt{2}}{2} L \end{cases}
\]  

(50)

The only solution that makes sense in our domain is \( x = \frac{\sqrt{2}}{2} L \), and therefore maximum occupancy is:

\[
O_{max} = O\left( \frac{\sqrt{2}}{2} L \right) = \frac{H\lambda w}{2L^2} \cdot \frac{L^4 - L^4}{4} = \frac{H\lambda wL^2}{8}
\]  

(51)

In vehicle travel time is derived as:

\[
IVTT = \frac{2}{\Lambda H} \int_0^L O(x) \cdot \frac{k(x)}{v_c(x)} \, dx
\]

\[
= \frac{2}{L^4} \int_0^L \left( x^2L^2 - x^4 \right) \cdot \left( \frac{1}{v} + \frac{1}{s} \cdot \frac{v}{a} + \lambda wH \cdot \tau' \right) \, dx 
\]

\[
= \frac{2L}{15} \left( \frac{1}{v} + \frac{v}{sa} \right) + \frac{L^2}{6} \cdot \lambda wH \tau' \quad \text{for } \alpha = 0
\]  

(52)

\[
IVTT = \frac{2}{\Lambda H} \int_0^L O(x) \cdot \frac{k(x)}{v_c(x)} \, dx
\]

\[
= \frac{2}{L^4} \int_0^L \left( x^2L^2 - x^4 \right) \cdot \left( 1 + \frac{\lambda wH}{3} \cdot x \right) \cdot \left( \frac{1}{v} + \frac{3\lambda wHx}{3 + \lambda wHx} \left( \frac{v}{a} + \tau' \right) \right) \, dx \quad \text{for } \alpha = 1
\]  

(53)
Average access time is only different to zero for conventional case:

\[
A = \frac{1}{2\nu_w \Lambda} \int_0^L \lambda w x \cdot [s + w] \, dx = \frac{s + w}{2\nu_w} \quad \text{for } \alpha = 0
\]

\[
A = 0 \quad \text{for } \alpha = 1
\]

Waiting time is constant but different depending on the service:

\[
W = \frac{1}{\Lambda} \int_0^L \lambda w x \cdot \omega \, dx = \begin{cases} \frac{H}{2} & \text{if } H < H_s \\ h_s + \frac{h_s}{h_s + f_s H} & \text{if } H \geq H_s \end{cases} \quad \text{for } \alpha = 0
\]

\[
W = \frac{1}{\Lambda} \int_0^L \lambda w x \cdot h_s \, dx = h_s \quad \text{for } \alpha = 1
\]

Annex 1: Deduction of extra detour of DRT due to one-directional street network

In this annex it is proved how the configuration of the street network affects the distance between two points respect to the Manhattan metric:

\[
D_{p,q \mid L1} = |x_p - x_q| + |y_p - y_q|
\]

It is shown that $L1$ is almost exact for a bidirectional street network and an alternative $L1^*$ metric is defined for the one-directional case.

Imagine an infinite uniform orthogonal grid, with constant street spacing $D_s$ and squared street blocks. Any trip from a point on the network to another consists of initial and final movements from the nearer/optimal crossroad to the exact start and end coordinates, plus a route between two crossroads.

Let’s focus on this second component as it is the one that dominates route length in most cases. If streets are bidirectional, the distance between crossroads is equal to the Manhattan distance for the same axis configuration. Therefore, there is no need to add any “extra” detour except for the extra distance that Manhattan metric already adds respect to Euclidean metric.

In the case of one-directional network though, this is not the case. We assume that direction changes regularly every other street. For a fixed initial crossroad point, every diagonal direction (NE, SE, SW and NW) has a subset of preferred streets that conform a wider grid, with spacing $2D_s$. For any of the 4 directions, trips inside this preferential network do not need any extra detour respect to Manhattan distance.
Therefore, the initial crossroad point can either be on this preferential network (¾ of the cases; NW, NE and SE in the image) or off it (¼ of the cases, SW). If the point happens to be off the network, it needs to travel one $D_s$ in order to reach it and another $D_s$ to compensate the deviation of reaching this preferential grid. The same happens with the final crossroad points, with ¾ being on the preferential network and ¼ being off. Then, the average extra detour regarding trips between crossroads is:

$$\Delta d_1 = \frac{1}{4} \cdot 2D_s + \frac{1}{4} \cdot 2D_s = D_s$$  \hspace{1cm} (59)$$

Now we can get back to the first component of the trips, the distance from the real point to the crossroad. In the bidirectional network, this distance is already taken into account by using L1 distance from the real start and end points, with the only exceptional cases of having origin and destination on the same columns or row of street blocks. Note that these exceptional cases will be less significant the larger the network; so for the definition of the metric in an infinite space they are neglected.

For the one-directional grid, this distance is already taken into account by L1 distance only in some of the cases. To find these detours we can still think using the preferential network scheme with the whole streets instead of only the crossroads. In this case, ½ of the points start on the preferential network and the other ½ off it. The same happens with the end points. In the case the point is already on the preferential network, the distance to reach its crossroad is already taken into account by the L1 measure. Nevertheless, if a point is off the preferential network, the L1 distance underestimates the real trip length as the vehicle needs to locally travel further away from the destination in order to reach the crossroad $\frac{D_s}{2}$ and to compensate this distance. Therefore, we can compute the average extra detour regarding the access to the crossroad as:

$$\Delta d_2 = \frac{1}{2} \cdot 2 \cdot \frac{D_s}{2} + \frac{1}{2} \cdot 2 \cdot \frac{D_s}{2} = D_s$$  \hspace{1cm} (60)$$

Finally if we include these effects inside the standard Manhattan metric we can define the distance between two points as:

$$D_{p,q} | L1^* = |x_p - x_q| + |y_p - y_q| + 2D_s$$ \hspace{1cm} (61)$$
Annex 2: Deduction of optimal band width $w^*$ and horizontal circuitry $k$ for the conventional service 0

In this annex, a more detailed explanation for the optimal band width and how it affects circuitry is explained.

For the conventional bus case, if a maximum access distance $r$ is defined, the threshold between a narrow (no stripes) and a wide (with vertical sweeping) corridor is defined at $w(x) = 2r - s(x)$, as the furthest point from a stop will be located at a distance of $\frac{s}{2} + \frac{w}{2}$, as seen in Figure 8.

Note also that for cases where $s \geq 2r$ there is not any feasible solution, therefore it must be set as a boundary condition of the optimization problem.

![Figure 8: Furthest point to a stop in a narrow corridor.](image)

The value for the optimal band width can be defined using the same logic, resulting in:

$$ w^*(x) = \begin{cases} w(x) & \text{if } w(x) \leq 2r - s(x) \\ 2r - s(x) & \text{if } w(x) > 2r - s(x) \end{cases} $$

(62)

Regarding circuitry, for every optimal band, of a width (horizontal) of $w^*$, a distance of $w$ is travelled. This can be visualised in Figure 9.

![Figure 9: Visualization for the formulation of $k(x)$.](image)

$$ k(x) = 1 + \frac{w(x) - w^*(x)}{w^*(x)} = \frac{w(x)}{w^*(x)} $$

(63)

Unfortunately, defining band width as $2r - s(x)$ does not always represent an optimal coverage of the area. In Figure 10, the coverage areas (in metric L1) for every stop are shown. As seen in the figure, the previously considered band width (standard) can be quite narrow as it only counts on the stops from the same vertical stripe to cover the area. Alternatively, if using a diamond
grid pattern, stops from the adjacent stripes can cover part of the middle areas, obtaining wider bands.

For the diamond grid case, band width is $w^* = r + \frac{2r-s}{2} = 2r - \frac{s}{2}$, and therefore formulation can be changed to:

$$w^*(x) = \begin{cases} w(x) & \text{if } w(x) \leq 2r - s(x) \\ \frac{2r - s(x)}{2} & \text{if } w(x) > 2r - s(x) \end{cases} \quad (64)$$

The downside of the diamond grid pattern is that the vehicle must reach the end of the area in order to ensure that coverage is complete, and therefore circuitry:

$$k(x) = \begin{cases} 1 & \text{if } w(x) \leq 2r - s(x) \\ \frac{w(x)}{w^*(x)} & \text{if } w(x) > 2r - s(x) \end{cases} \quad (65)$$

Therefore, in the wide corridor scenario, if the corridor is not extremely wide, the conventional strategy is optimal, but for very wide corridors, it is preferred to implement diamond grid coverage. The changing point occurs at:

$$w_d(x) = \frac{2}{s(x)} \cdot (2r - s(x)) \cdot \left(2r - \frac{s(x)}{2}\right) \quad (66)$$

We can therefore combine the two strategies as:

$$w^*(x) = \begin{cases} w(x) & \text{if } w(x) \leq 2r - s(x) \\ 2r - s(x) & \text{if } 2r - s(x) < w(x) \leq w_d(x) \\ \frac{2r - s(x)}{2} & \text{if } w(x) > w_d(x) \end{cases} \quad (67)$$

$$k(x) = \begin{cases} 1 & \text{if } w(x) \leq 2r - s(x) \\ \frac{w(x)}{w^*(x)} & \text{if } 2r - s(x) < w(x) \leq w_d(x) \\ 1 + \frac{w(x)}{w^*(x)} & \text{if } w(x) > w_d(x) \end{cases} \quad (68)$$
REFERENCES


